Distributed control for consensus on leader-followers proximity graphs

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Outline:

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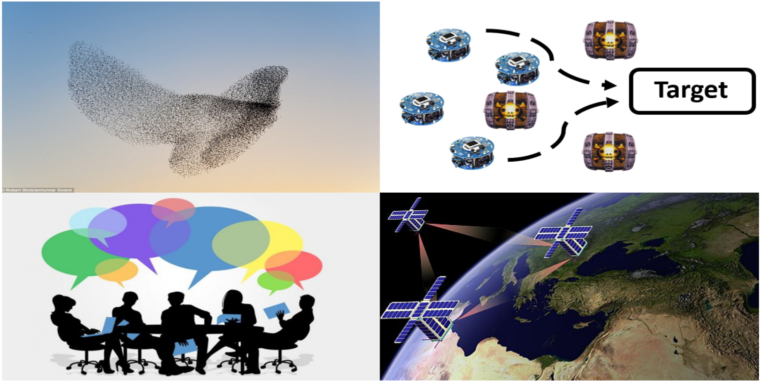
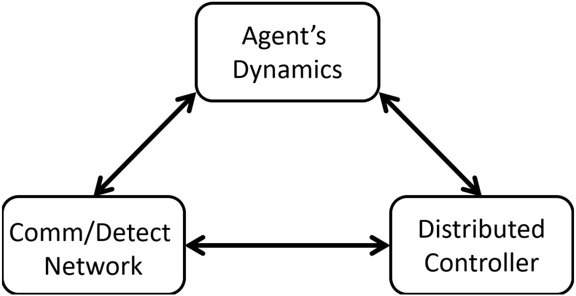
Collective behaviors

In nature, when big groups of individuals jointly operate, exhibit auto-organized behaviors (*e.g.* flocking, synchronization and consensus).



Multiagent system (MAS)

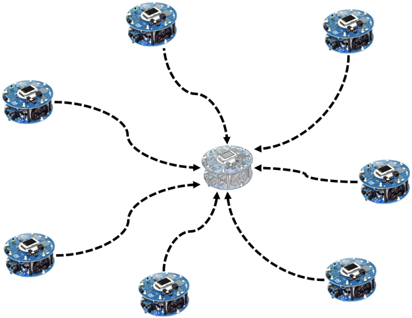
MAS, consists on a group of dynamic subsystems, called agents, interacting with each other on local neighborhoods through communication links and/or local sensing, sharing their local state, and using the collected information to update it's state according to a distributed controller*.



*K. Sakurama, S. Azuma and T. Sugie, "Distributed Controllers for Multi-Agent Coordination Via Gradient-Flock Approach", *IEEE Trans. on Auto. Control*, 2015.

Consensus on MAS

The group of agents reach an agreement on their local variables. The final common value is called a *consensus state*[*].



[*] R. Olfati-Saber and R.M. Murray, "Consensus Problems in Networks of Agents with Switching Topology and Time-Delays", *IEEE Trans. on Auto. Control*, 2004.

Outline:

Agent's dynamics

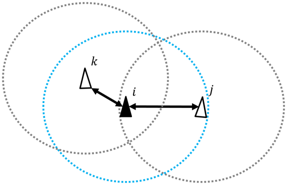
Consider a group of N inertial agents with dynamics

$$\dot{p}_i = v_i, \quad m_i \dot{v}_i = u_i, \quad i = 1, \cdots, N, \tag{1}$$

where $p_i, v_i, u_i \in \mathbb{R}^n$ and $m_i \in \mathbb{R}_{>0}$. Also, suppose all agents have the same sensing/communication radio $r \in \mathbb{R}_{>0}$.

Proximity graph

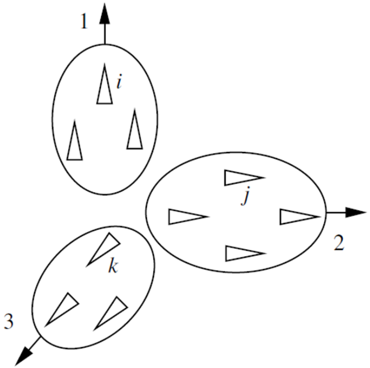
Is a graph $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$ consisting of a nodes set $\mathcal{V} = \{1, \cdots, N\}$ and a position dependent time varying set of edges $\mathcal{E}(t) = \{(i,j) | i,j \in \mathcal{V}\}$.



$\mathbb{R}^n$ states for the n-dimensional real vector set, $\mathbb{R}_{>0}$ for all positive real values.

Control objectives: Connectivity preservation

When network links depends on relative positions, a common pitfall is the *fragmentation phenomenon*[*].



[*] R. Olfati-Saber "Flocking for multi-agent dynamic systems: Algorithms and theory ", *IEEE Trans. on Auto. Control*, Vol. 51, 2006.

Control objectives: Leader following

Consider the desired common value is defined by a virtual leader's dynamics is

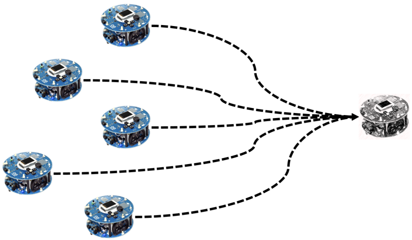
$$\dot{p}_l = v_l, \quad \dot{v}_l = f\left(p_l, v_l, t\right), \tag{2}$$

where $p_l, v_l \in \mathbb{R}^n$ and $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_{\geq 0} \mapsto \mathbb{R}^n$ is a continuous Lipschitz function.

Leader-followers consensus problem

A leader-followers consensus is achieved if, for any admissible initial conditions,

$$\lim_{t \to \infty} \|p_i - p_l\| = 0 \text{ and } \lim_{t \to \infty} \|v_i - v_l\| = 0, \quad i = 1, \cdots, N \tag{3}$$



Outline:

Distributed controller for constant velocity leader

Assumption

The virtual leader moves at a constant velocity, *i.e.* $f(p_l, v_l, t) \equiv 0$ in (2).

Consider the following distributed controller

$$u_i = -\sum_{j\in\mathcal{N}_i} \nabla_{p_i}\psi\left(\|p_{ij}\|\right) - \sum_{j\in\mathcal{N}_i} a_{ij}(v_i - v_j) - h_i\left((p_i - p_l) + (v_i - v_l)\right), \qquad (4)$$

where

- $\nabla_{p_i}\psi\left(\|p_{ij}\|\right)$ is an Artificial potential function (APF) gradient respect to p_i;
- a_{ij} is the ij-th element of adjacency matrix $\mathcal{A}(\mathcal{G}(t))$;
- $\mathcal{N}_i = \left\{j \in \mathcal{V} | (i,j) \in \mathcal{E}(t)\right\}$ is the neighbors set of agent i;
- $h_i \in \mathbb{R}_{>0}$ if agent i receives information from the leader and $h_i = 0$ otherwise.

$\|\cdot\|$ is the Euclidean norm and $p_{ij} = p_i - p_j$.

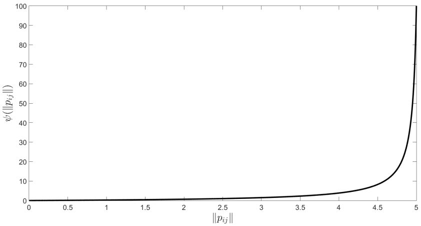
Artificial potential function (APF)

Consider a nonnegative potential function such that depends on relative distances between agents $\|p_{ij}\|$, differentiable for $\|p_{ij}\| \in [0, r]$ and satisfying

(i) $\psi(\|p_{ij}\|) \to \bar{\psi}$ as $\|p_{ij}\| \to r$;

(ii) $\frac{\partial\psi(\|p_{ij}\|)}{\partial\|p_{ij}\|} > 0$ for $\|p_{ij}\| \in (0, r)$;

(iii) $\lim_{\|p_{ij}\|\to 0}\left(\frac{\partial\psi(\|p_{ij}\|)}{\partial\|p_{ij}\|}\frac{1}{\|p_{ij}\|}\right)$ is nonnegative and bounded.



An example*:

$$\psi(\|p_{ij}\|) = \frac{\bar{\psi}\,\|p_{ij}\|^2}{\bar{\psi}(r - \|p_{ij}\|) + \|p_{ij}\|^2}$$

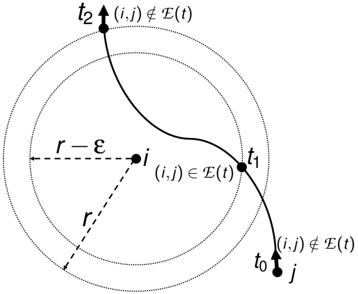
* H. Su, X. Wang and G. Chen "Rendezvous of Multiple Mobile Agents with Preserved Network Connectivity ", *Sys. Control Lett*, Vol. 59, 2010.

Dynamic set of links: An hysteresis process

The set $\mathcal{E}(t)$ evolves accordingly to the following process:

1. initial links are $\mathcal{E}(t_0) = \{(i,j) | \, \|p_{ij}(t_0)\| < r - \varepsilon\}$, for every $i,j \in \mathcal{V}$;
2. if link $(i,j) \notin \mathcal{E}(t^-)$ and $\|p_{ij}\| < r - \varepsilon$, then $(i,j) \in \mathcal{E}(t)$ and;
3. if $\|p_{ij}\| \geq r$, then $(i,j) \notin \mathcal{E}(t)$.

where $\varepsilon \in (0, r)$ and t^- is the instant before t.



M. Ji and M. Egerstedt, "Distributed Coordination Control of Multiagent Systems While Preserving Connectedness ", *IEEE Trans. Robot.*, Vol. 23, 2007.

Result for leader with constant velocity: $f(p_l, v_l, t) \equiv 0$

Theorem 1

Consider a system of N inertial agents with model (1) applying controller (4) and a virtual leader with dynamics (2) with $f(p_l, v_l, t) \equiv 0$. Suppose the initial proximity graph $\mathcal{G}(0)$ is connected, and the initial error conditions $\tilde{p}(0), \tilde{v}(0) \in \Omega_0$, then the following results hold:*

(i) *$\mathcal{G}(t)$ remains connected all time $t \geq 0$,*

(ii) *all agents asymptotically converge to leader's position and velocity.*

Proof sketch†

(i) Define a function $V(\tilde{p}, \tilde{v}) \leq \bar{V}(\tilde{p}(0), \tilde{v}(0)) < \bar{\psi}$ with time derivative $\dot{V}(\tilde{v}) \leq 0$.

(ii) Using LaSalle's invariance principle, a set such that $V(\tilde{p}, \tilde{v}) \leq \bar{V}(\tilde{p}(0), \tilde{v}(0))$ is positively invariant with $\dot{V}(\tilde{v}) = 0$ iff $v_i = v_l$ for all $i \in \mathcal{V}$.

(iii) From controller (4) and APF's definition, position consensus $p_i = p_l$ is proved.

* $\Omega_0 = \{\tilde{p}(0) \in \mathbb{R}^{Nn}, \tilde{v}(0) \in \mathbb{R}^{Nn} : \bar{V}(\tilde{p}(0), \tilde{v}(0)) < \bar{\psi}\}$, where $\tilde{p} = [\tilde{p}_1^T, \cdots, \tilde{p}_N^T]^T$ and $\tilde{v} = [\tilde{v}_1^T, \cdots, \tilde{v}_N^T]^T$, with $\tilde{p}_i = p_i - p_l$ and $\tilde{v}_i = v_i - v_l$.

† A more general description is available for discussion at the end of presentation.

Example: Leader with constant velocity

Distributed controller for leader with time-varying velocity

Assumption

Leader's and agent's accelerations can be communicated or calculated through local sensing*.

Consider the following distributed controller

$$u_i = -\frac{1}{\eta_i}\sum_{j\in\mathcal{N}_i}\nabla_{p_i}\psi(\|p_{ij}\|) - \frac{1}{\eta_i}\sum_{j\in\mathcal{N}_i}a_{ij}(v_i - v_j) + \frac{1}{\eta_i}\sum_{j\in\mathcal{N}_i}a_{ij}\dot{v}_j$$
$$-\frac{h_i}{\eta_i}\left((p_i - p_l) + (v_i - v_l) - \dot{v}_l\right), \quad (5)$$

where $\eta_i = \frac{1}{m_i}\left(h_i + \sum_{j\in\mathcal{N}_i}a_{ij}\right)$, which for connected networks is always positive.

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* W. Ren and R.W. Beard "Distributed Consensus in Multi-vehicle Cooperative Control: Theory and Applications", *Springer-verlag*, 2008.

Result for leader with time-varying velocity

Theorem 2

Consider a system of N inertial agents with model (1) applying controller (5) and a virtual leader with dynamics (2). Suppose the initial proximity graph $\mathcal{G}(0)$ is connected, and the initial error conditions $\tilde{p}(0), \tilde{v}(0) \in \Omega_0{}^*$, then the following results hold:

(i) $\mathcal{G}(t)$ *remains connected all the time* $t \geq 0$*,*

(ii) *all agents asymptotically converge to leader's position and velocity.*

Proof sketch†

(i) Define a function $W(\tilde{p}, \tilde{v}) \leq \bar{W}(\tilde{p}(0), \tilde{v}(0)) < \bar{\Psi}$ with time derivative $\dot{W}(\tilde{v}) \leq 0$.

(ii) This proof follows the same steps has Theorem's 1 proof.

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* $\Omega_0 = \{\tilde{p}(0) \in \mathbb{R}^{Nn}, \tilde{v}(0) \in \mathbb{R}^{Nn} : \bar{W}(\tilde{p}(0), \tilde{v}(0)) < \bar{\Psi}\}$

† A more general description is available for discussion at the end of presentation.

Example: Leader with time-varying velocity

Outline:

1. Introduction
2. Problem Statement
3. Distributed Controllers
4. **Final Comments**

Final comments

Summary

- Leader-followers consensus problem over proximity graphs is investigated.
- Two fully distributed controllers were developed; The first, considering the leader moves at a constant velocity; The second, for time-varying leader's velocity.
- This results extends the work made by Su *et. al.* 2010*, where leader-followers consensus is also investigated (just for leader's velocity), and all agents have access to leader's acceleration.

Future work

- Collective behaviors problems on MAS with different sensing radio for each agents while avoid collisions with environmental obstacles.
- Implement distributed controllers in groups of mobile robots (for consensus and flocking).

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* H. Su, X. Wang and G. Chen "Rendezvous of multiple mobile agents with preserved network connectivity ", *Syst. Control Lett.*, Vol. 59, 2010.

Thank you all !!

Questions?

Outline:

Some graph theory

Adjacency matrix $\mathcal{A}(\mathcal{G}(t)) \in \mathbb{R}^{N\times N}$:

$$a_{ij} = \begin{cases} a_{ij} \in \mathbb{R}_{>0}, & \text{if } (i,j) \in \mathcal{E}(t), \\ 0, & \text{otherwise,} \end{cases}$$

with diagonal elements $a_{ii} = 0$.

Laplacian matrix $\mathcal{L}(\mathcal{G}(t)) \in \mathbb{R}^{N\times N}$:

$$l_{ij} = \begin{cases} l_{ij} = -a_{ij}, & \text{if } (i,j) \in \mathcal{E}(t), \\ 0, & \text{otherwise,} \end{cases}$$

with diagonal elements $l_{ii} = \sum_{j=1, j\neq i} a_{ij}$.

A graph $\mathcal{G}(t)$ is connected if there exists a path (a sequence of edges $(i,j) \in \mathcal{E}(t)$) connecting every pair of nodes. Additionally, it's Laplacian satisfies*

$$z^T(\mathcal{L}\otimes I_n)z = \frac{1}{2}\sum_{(i,j)\in\mathcal{E}(t)} a_{ij}\left\|z_i - z_j\right\|^2, \tag{6}$$

where $z = \left[z_1^T, \cdots, z_N^T\right]^T \in \mathbb{R}^{nN}$ with $z_i \in \mathbb{R}^n$, I_n is the n-dimensional identity matrix and $\otimes$ is the Kronecker product.

* R. Olfati-Saber "Flocking for multi-agent dynamic systems: Algorithms and theory ", *IEEE Trans. on Auto. Control*, Vol. 51, 2006.

Outline:

Theorem 1: Candidate function

Let $\tilde{p}_i = p_i - p_l$ and $\tilde{v}_i = v_i - v_l$ be state errors, and define the following function

$$V(t) = \frac{1}{2}\sum_{i=1}^{N}\left(\sum_{j\in\mathcal{N}_i}\psi\left(\|\tilde{p}_{ij}\|\right) + h_i\tilde{p}_i^T\tilde{p} + m_i\tilde{v}_i^T\tilde{v}_i\right); \tag{7}$$

The initial energy of the complete system $V_0 = (p(0), v(0))$ is bounded, since

$$V_0 \leq \frac{1}{2}\sum_{i=1}^{N}\left(m_i\tilde{v}_i^T(0)\tilde{v}_i(0) + h_i\tilde{p}_i^T(0)\tilde{p}_i(0)\right) + \frac{N(N-1)}{2}\psi\left(r-\varepsilon\right) = \bar{V} \tag{8}$$

Also, define the set $\Omega_0 = \left\{\tilde{p}(0), \tilde{v}(0) \in \mathbb{R}^{nN} : \bar{V} < \bar{\psi}\right\}$. Notice, error dynamics is

$$\dot{\tilde{p}}_i = \tilde{v}_i, \quad m_i\dot{\tilde{v}}_i = u_i, \quad i = 1, \ldots, N, \tag{9}$$

where u_i can be rewritten on errors terms

$$u_i = -\sum_{j\in\mathcal{N}_i}\nabla_{\tilde{p}_i}\psi\left(\|\tilde{p}_{ij}\|\right) - \sum_{j\in\mathcal{N}_i}a_{ij}(\tilde{v}_i - \tilde{v}_j) - h_i\left(\tilde{p}_i + \tilde{v}_i\right) \tag{10}$$

Proof sketch: Connectivity preservation

Assume network switches on instants t_k with $k = 1, 2, \dots$ and remains fixed over interval $[t_{k-1}, t_k)$. Taking time derivative of (7) yields

$$\dot{V}(t) = \sum_{i=1}^{N} \left(\frac{1}{2} \sum_{j \in \mathcal{N}_i} \dot{\psi}(\|\tilde{p}_{ij}\|) + h_i \dot{\tilde{p}}_i^T \tilde{p}_i + m_i \tilde{v}_i^T \dot{\tilde{v}}_i \right) = -\tilde{v}^T (\mathcal{L}_{\mathcal{H}} \otimes I_n) \tilde{v} \leq 0 \qquad (11)$$

where $\mathcal{L}_{\mathcal{H}} = \mathcal{L} + \mathcal{H}$ with $\mathcal{H} = diag(h_1, \dots, h_N)$. Equation (11), implies that

- (i) since $\tilde{p}(0), \tilde{v}(0) \in \Omega_0$, then $V(t) \leq \bar{V} < \bar{\psi}$ for $t \in [t_0, t_1)$, thus no distance $\|p_{ij}\| \to r$. Then, on t_1 some edges are added to $\mathcal{G}(t)$;
- (ii) Assume there are $0 < q_1 \leq \frac{(N-1)(N-2)}{2}$ new edges on t_1, thus $V(t_1) \leq V_0 + q_1 \psi(r - \varepsilon) \leq \bar{V} < \bar{\psi}$;
- (iii) Applying recursively the aforementioned analysis, we conclude that $\mathcal{G}(t)$ remains connected for all $t \geq 0$.

Proof sketch: Consensus with leader (Velocity)

From the aforementioned analysis notice:

- Number of new edges is finite $0 < q_k \leq \frac{(N-1)(N-2)}{2}$, thus $\mathcal{G}(t)$ gets fixed;
- The set $\Omega = \left\{ \hat{\tilde{p}} \in D_{\mathcal{G}}, \tilde{v} \in \mathbb{R}^{nN} : V(\hat{\tilde{p}}, \tilde{v}) \leq \bar{V} \right\}$ is positively invariant, where $D_{\mathcal{G}} = \left\{ \hat{\tilde{p}} \in \mathbb{R}^{nN^2} : \|\tilde{p}_{ij}\| \in \left[0, \psi^{-1}(\bar{V})\right], \forall (i,j) \in \mathcal{E}(t) \right\}^*$;
- From LaSalle's invariance principle, all trajectories converge to $S = \left\{ \hat{\tilde{p}} \in D_{\mathcal{G}}, \tilde{v} \in \mathbb{R}^{nN} : \dot{V} = 0 \right\}$;
- From (11), notice that $\dot{V}(t) = -\tilde{v}^T (\mathcal{L} \otimes I_n) \tilde{v} - \tilde{v}^T (\mathcal{H} \otimes I_n) \tilde{v} = 0$, implying $\tilde{v}_1 = \ldots = \tilde{v}_N$ and $\tilde{v}_i = 0$ for any i such that $h_i > 0$, i.e. $v_1 = \ldots = v_N = v_l$;

* with $\hat{\tilde{p}} = \left[\tilde{p}_{11}^T, \ldots, \tilde{p}_{1N}^T, \ldots, \tilde{p}_{N1}^T, \ldots, \tilde{p}_{NN}^T\right]^T$

Proof sketch: Consensus with leader (Position)

In steady state $\dot{\tilde{v}}_i = 0$, thus from controller (4) we have

$$u_i = -\sum_{j \in \mathcal{N}_i} \frac{\partial \psi\left(\|\tilde{p}_{ij}\|\right)}{\partial \|\tilde{p}_{ij}\|} \frac{\tilde{p}_i - \tilde{p}_j}{\|\tilde{p}_{ij}\|} - h_i \tilde{p}_i = 0_n \tag{12}$$

rewriting the last equation in a matrix form for all agents and multiplying by $\tilde{p}^T$

$$-\tilde{p}^T \left(\hat{\mathcal{L}} \otimes I_n\right) \tilde{p} - \tilde{p}^T \left(\mathcal{H} \otimes I_n\right) \tilde{p} = 0 \tag{13}$$

where

$$\hat{\mathcal{L}}_{ii} = \sum_{j=1, j \neq i}^{N} \left(\frac{\partial \psi\left(\|\tilde{p}_{ij}\|\right)}{\partial \|\tilde{p}_{ij}\|} \frac{1}{\|\tilde{p}_{ij}\|} \right) \quad \text{and} \quad \hat{\mathcal{L}}_{ij} = -\frac{\partial \psi\left(\|\tilde{p}_{ij}\|\right)}{\partial \|\tilde{p}_{ij}\|} \frac{1}{\|\tilde{p}_{ij}\|} \quad \text{for } i \neq j,$$

which implies that $\tilde{p}_1 = \ldots = \tilde{p}_N$ and $\tilde{p}_i = 0$ for any i such that $h_i > 0$, i.e. $p_1 = \ldots = p_N = p_l$.

Outline:

Theorem 2: Candidate function

Define the next function

$$W(t) = \frac{1}{2}\sum_{i=1}^{N}\left(\sum_{j\in\mathcal{N}_i}\psi\left(\|\tilde{p}_{ij}\|\right) + h_i\tilde{p}_i^T\tilde{p}_i\right) + \frac{1}{2}\tilde{v}^T\left(\mathcal{L}_{\mathcal{H}}\otimes I_n\right)\tilde{v} \tag{14}$$

The initial energy of the complete system $W_0 = W(\tilde{p}(0), \tilde{v}(0))$ is bounded on the next way

$$W_0 \leq \frac{N(N-1)}{2}\psi\left(r-\varepsilon\right) + \frac{1}{2}\sum_{i=1}^{N}h_i\tilde{p}_i^T(0)\tilde{p}_i(0) + \frac{1}{2}\tilde{v}^T(0)\left(\mathcal{L}_{\mathcal{H}}\otimes I_n\right)\tilde{v}(0) = \bar{W}$$

Also, define the initial conditions set $\Omega_0 = \left\{\tilde{p}(0), \tilde{v}(0) \in \mathbb{R}^{nN} : \bar{W} < \bar{\psi}\right\}$. The error dynamics is

$$\dot{\tilde{p}}_i = \tilde{v}_i, \quad m_i\dot{\tilde{v}}_i = u_i - m_i\dot{v}_l, \quad i = 1, \ldots, N. \tag{15}$$

Controller (5) can be rewritten in therms of error states like

$$u_i = -\frac{1}{\eta_i}\sum_{j\in\mathcal{N}_i}\nabla_{\tilde{p}_i}\psi\left(\|\tilde{p}_{ij}\|\right) - \frac{1}{\eta_i}\sum_{j\in\mathcal{N}_i}a_{ij}(\tilde{v}_i - \tilde{v}_j) + \frac{1}{\eta_i}\sum_{j\in\mathcal{N}_i}a_{ij}\dot{v}_j - \frac{h_i}{\eta_i}\left(\tilde{p}_i + \tilde{v}_i - \dot{v}_l\right). \tag{16}$$

After some manipulations, error dynamics (15) with controller (16), results on

$$\begin{aligned} \dot{\tilde{p}}_i =& \tilde{v}_i, \\ \sum_{j\in\mathcal{N}_i} a_{ij}(\dot{\tilde{v}}_i - \dot{\tilde{v}}_j) + h_i\dot{\tilde{v}}_i =& -\sum_{j\in\mathcal{N}_i} \nabla_{\tilde{p}_i}\psi\left(\|\tilde{p}_{ij}\|\right) - h_i\tilde{p}_i - \sum_{j\in\mathcal{N}_i} a_{ij}(\tilde{v}_i - \tilde{v}_j) - h_i\tilde{v}_i. \end{aligned}$$

Rewriting last equation on a more compact form we have

$$\begin{aligned} \dot{\tilde{p}} &= \tilde{v}, \\ \left(\mathcal{L}_{\mathcal{H}} \otimes I_n\right)\dot{\tilde{v}} &= -\left(\hat{\mathcal{L}}_{\mathcal{H}} \otimes I_n\right)\tilde{p} - \left(\mathcal{L}_{\mathcal{H}} \otimes I_n\right)\tilde{v}. \end{aligned} \quad (17)$$

Since

$$\dot{W}(t) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i}\dot{\psi}(\|\tilde{p}_{ij}\|) + \sum_{i=1}^{N} h_i\dot{\tilde{p}}_i^T\tilde{p}_i + \tilde{v}^T\left(\mathcal{L}_{\mathcal{H}} \otimes I_n\right)\dot{\tilde{v}} = -\tilde{v}^T\left(\mathcal{L}_{\mathcal{H}} \otimes I_n\right)\tilde{v} \leq 0, \quad (18)$$

this theorem can be proved following the sames steps as in theorem 1.